CSE331 Introduction to Algorithms
Lecture 13
Optimal Binary Search Trees

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3. Optimal binary search trees
4. Structure of an Optimal BST
5. Recursive Solution
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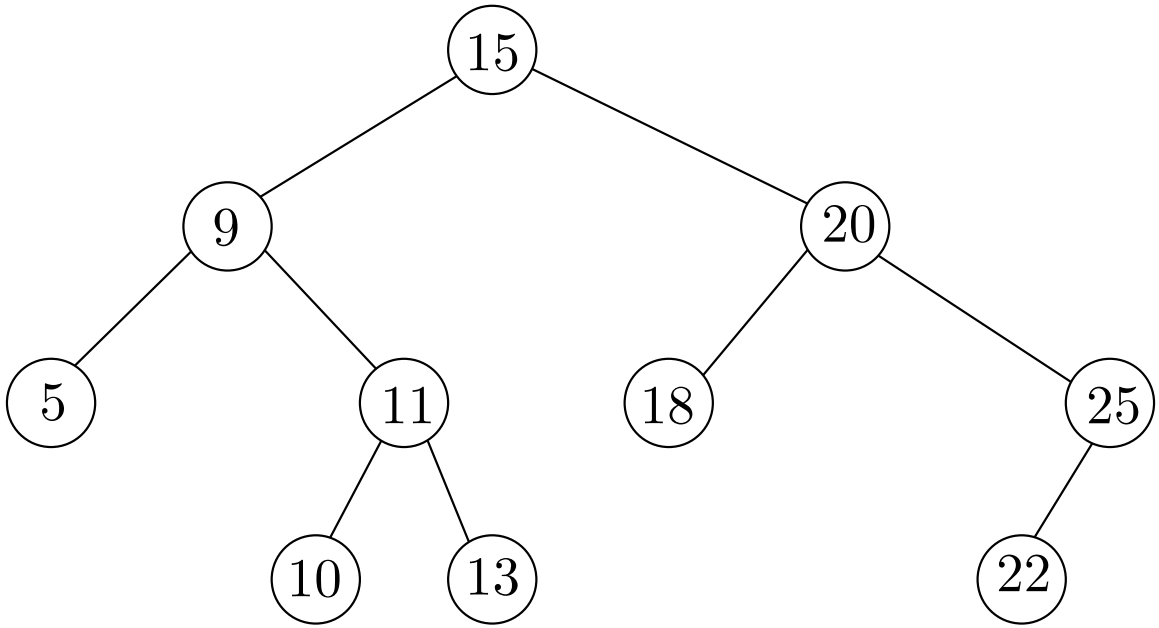
Introduction

- In this lecture, we will show how to compute optimal *binary search trees*.
- We will solve it using dynamic programming.
- The lecture will start a short review of binary search trees.
- **Reference**: Section 12 (p. 286) and Section 15.5 of the textbook (p. 397)
  Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.

Binary Search Trees

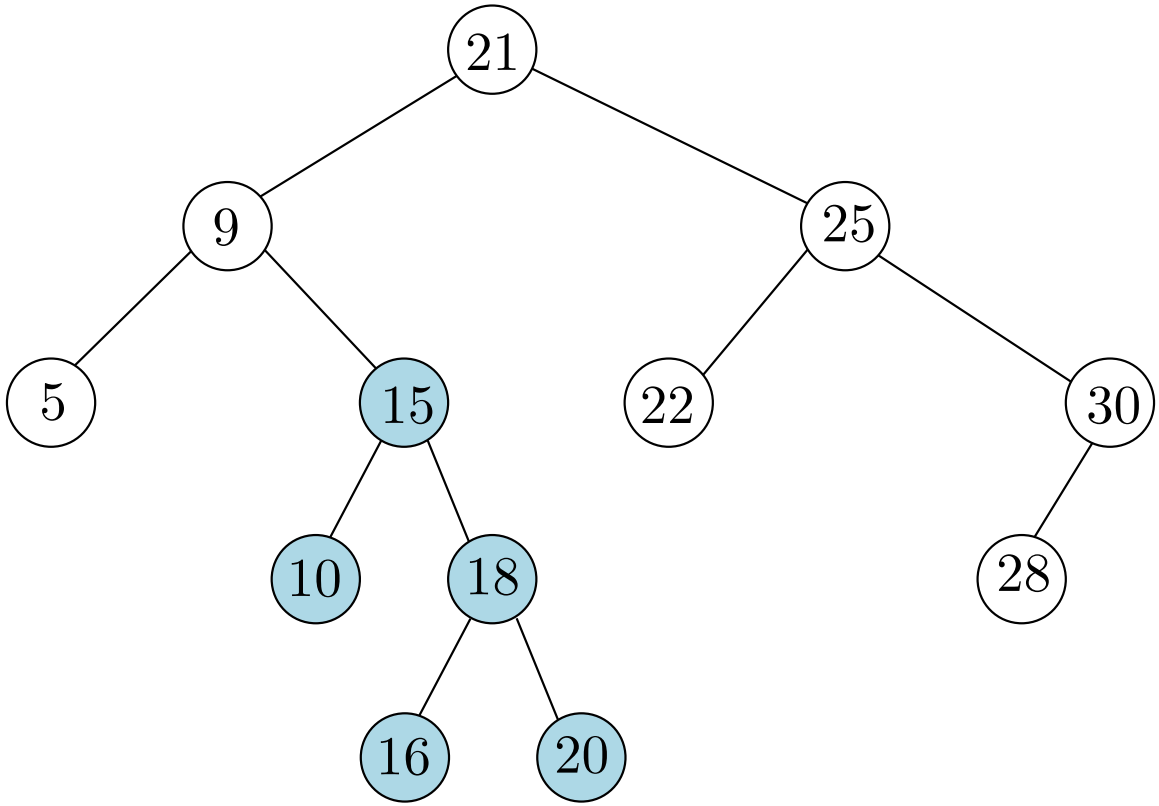


Definition (Binary search tree)

A *binary search tree (BST)* T is a binary tree that records at each node v a key $\text{key}(v)$. Every node v of T has the following properties.

- For every node u in the left subtree of v, we have $\text{key}(u) \leqslant \text{key}(v)$.
- For every node w in the right subtree of v, we have $\text{key}(w) \geqslant \text{key}(v)$.

Subtrees of a BST



- BST with set of keys $\{5, 9, 10, 15, 16, 18, 20, 21, 22, 25, 28, 30\}$.

Subtrees of a BST

Proposition

The keys stored in a subtree T' of a binary search tree T are consecutive. So if the keys of T are $k_1 < k_2 < \cdots < k_n$, then T' records $k_i < k_{i+1} < \cdots < k_j$ for some $i, j \in \{1, \ldots, n\}$.

Proof.

Done in class. □

Binary Search Trees

Implementation

A node v of a BST records the following fields:

- key(v) — the key of v
- left(v) — pointer to the left child of v
- right(v) — pointer to the right child of v

The pointer left(v) or right(v) is set to NIL if the corresponding child does not exist.

- Node v may also record satellite data
- For instance, if T records points (x, y, z), the key could be x and (y, z) could be the satellite data.
- In this lecture we do not use satellite data.

Insertion into a BST

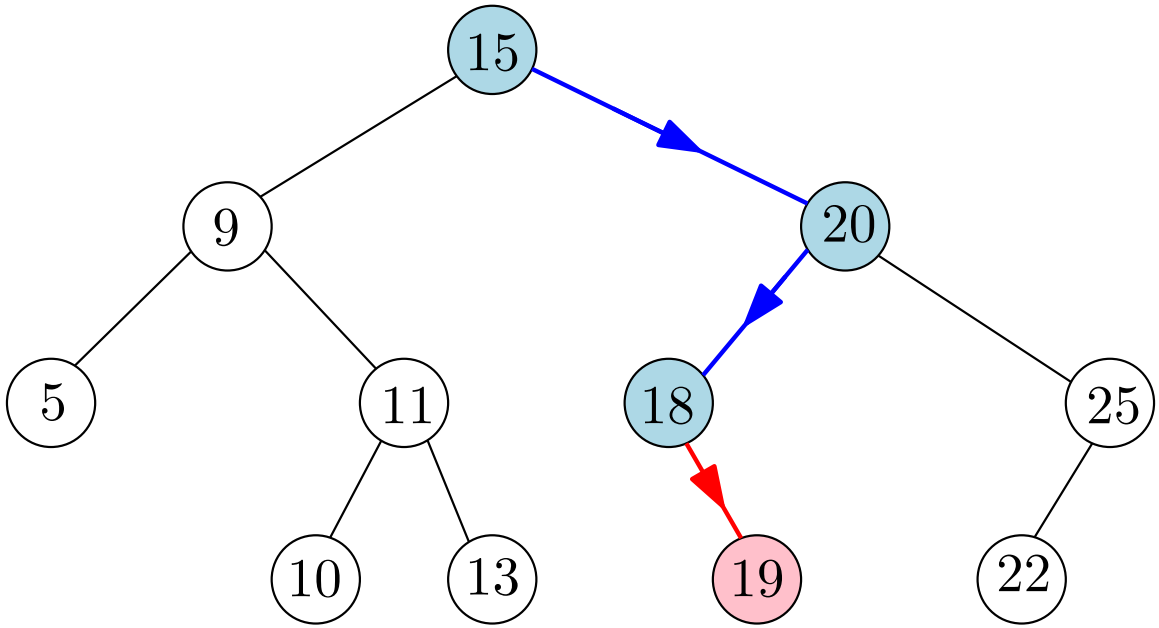
Inserting key k into a BST

```
1: procedure INSERT(r, k)
2:     if r = NIL then
3:         r ← NEWNODE(k)
4:     else if k < key(r) then
5:         INSERT(left(r), k)
6:     else
7:         INSERT(right(r), k)
```

- The new key k is inserted from the *root* node r of the tree T.
- The root node is the only node without parent.
- Insertion takes $O(h+1)$ time, where h is the height of the tree.

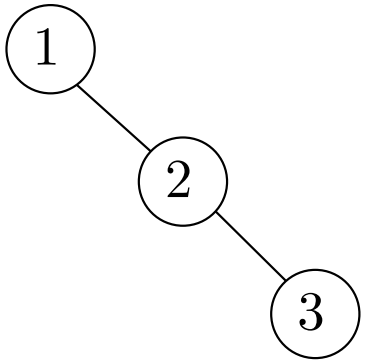
BST Insertion: Example

- Inserting 19 into the tree from Slide 4

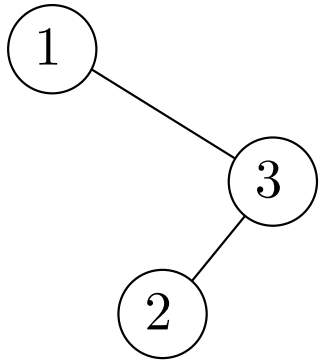


BST Insertion Orders

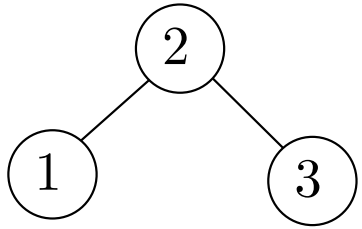
- The shape of a BST depends on the order of insertions.



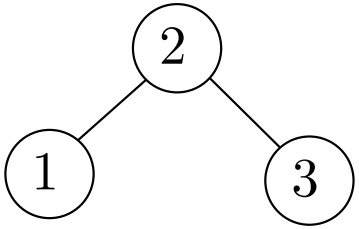
$1 \to 2 \to 3$



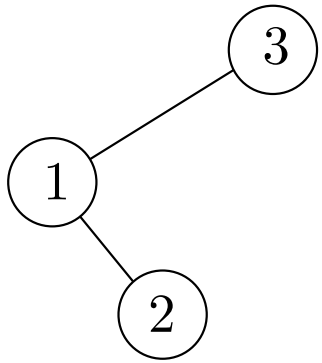
$1 \to 3 \to 2$



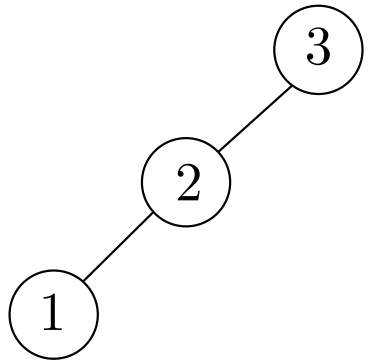
$2 \to 1 \to 3$



$2 \to 3 \to 1$



$3 \to 1 \to 2$



$3 \to 2 \to 1$

In-Order Traversal

- The keys of a binary search tree T can be printed in nondecreasing order by calling the following procedure, called *in-order traversal*, from the root of T.

Pseudocode

```
1: procedure IN-ORDER(v)
2:     if v = NIL then
3:         return
4:     IN-ORDER(left(v))
5:     Print key(v)
6:     IN-ORDER(right(v))
```

- On the BST from Slide 4, it prints:

5 9 10 11 13 15 18 20 22 25

Searching in a BST

Problem (Searching)

Given a binary search tree T and a key k, the searching problem is to decide whether k is the key of a node v of T, and if so, return v.

- The procedure on next slide allows to search in a BST in $O(h+1)$ time, where h is the height of the tree.

Searching in a BST

Pseudocode

```
1: procedure SEARCH(v, k)
2:     if v = NIL then
3:         return NOTFOUND
4:     if k < key(v) then
5:         return SEARCH(left(v), k)
6:     if k > key(v) then
7:         return SEARCH(right(v), k)
8:     return v                                  ▷ k = key(v)
```

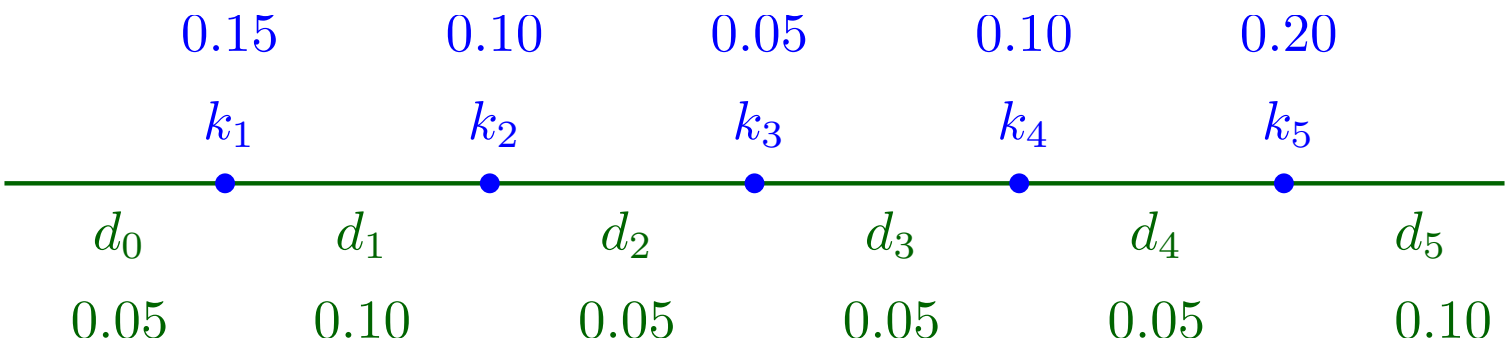
Balanced Binary Search Trees

- A BST with n nodes has height at least $\lfloor \log n \rfloor$, so the (worst case) search time is $\Omega(\log n)$.
- There exist *balanced binary search trees* whose height is $O(\log n)$, so the search time is $\Theta(\log n)$.
- It is also possible to insert and delete nodes in $\Theta(\log n)$ time in a balanced BST.
  - It requires to rebalance (change the structure) of the BST while inserting/deleting.
- So balanced BST have the same asymptotic search time as a sorted array, and allow efficient insertion/deletion. Sorted arrays, on the other hand, do not allow efficient insertion/deletion.
- Balanced binary search trees are not covered in CSE331.

  (Covered in CSE221 Data structures.)

Optimal Binary Search Trees

- We are given a set of keys $k_1 < k_2 < \cdots < k_n$.
- We want to build a BST on this set of keys, so as to answer search queries efficiently.
- We know in advance the probability distribution of the queries. Example:



i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Optimal Binary Search Trees

Input

- A list $k_1 < k_2 < \cdots < k_n$ of n keys.
- The probability p_i that a query falls on k_i, for $i = 1, \ldots, n$.
- The probability q_i that a query is between k_i and k_{i+1}, for $i = 1, \ldots, n$.
- The probability q_0 that a query is less than k_0.
- The probability q_n that a query is more than k_n.

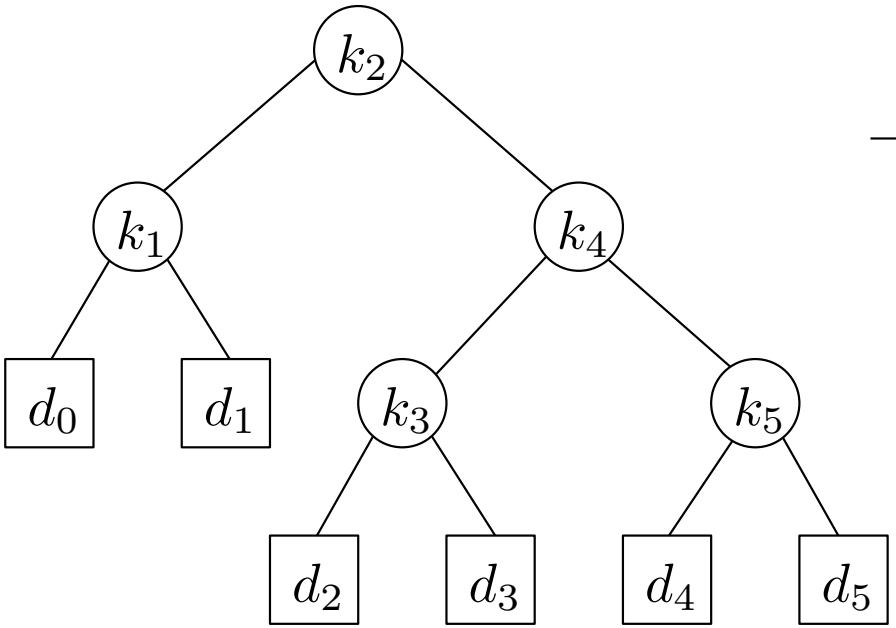
- The probabilities should sum to 1, so we have

$$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1.$$

Optimal Binary Search Trees

- Several different BST can record the keys.
- Goal: Construct a BST such that the expected query time is minimized.
- The BST will contain dummy keys d_i at the leaves, each d_i corresponding to the interval that has probability q_i.

Example



i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Input table

- Is it a good solution?
- No. Intuitively, we would like k_5 to be closer to the root, as it has the highest probability.

Problem Formulation

- The *depth* of a node is the length of the path from the root.
- We assume that the search time for a node v is proportional to $\text{depth}_T(v) + 1$, which is the number of nodes on the path from the root to v.
- Then the expected search time in a tree T is proportional to

$$\sum_{i=1}^{n} p_i(\text{depth}_T(k_i) + 1) + \sum_{i=0}^{n} q_i(\text{depth}_T(d_i) + 1)$$

$$= 1 + \sum_{i=1}^{n} p_i \,\text{depth}_T(k_i) + \sum_{i=0}^{n} q_i \,\text{depth}_T(d_i).$$

- So our goal is to construct a tree T that minimizes

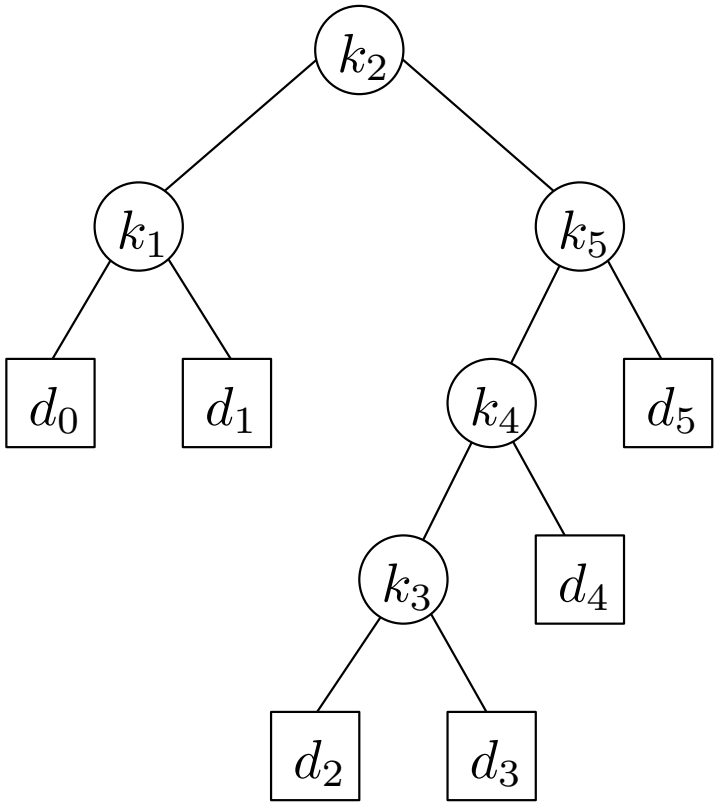
$$\text{cost}(T) = \sum_{i=1}^{n} p_i \,\text{depth}_T(k_i) + \sum_{i=0}^{n} q_i \,\text{depth}_T(d_i). \tag{1}$$

Example

node	depth	probability	contribution
k_1	1	0.15	0.15
k_2	0	0.10	0
k_3	2	0.05	0.10
k_4	1	0.10	0.10
k_5	2	0.20	0.40
d_0	2	0.05	0.10
d_1	2	0.10	0.20
d_2	3	0.05	0.15
d_3	3	0.05	0.15
d_4	3	0.05	0.15
d_5	3	0.10	0.30
Total			1.80

Table: Calculation of cost(T) for the tree T on Slide 18. According to our model, the expected search time will be proportional to $1 + 1.80 = 2.80$

Example



i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

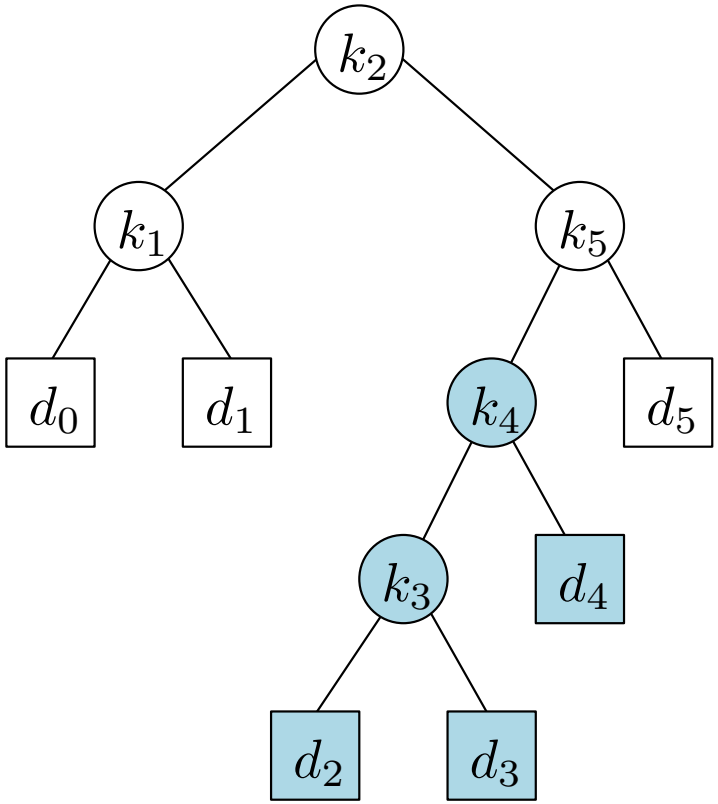
Input table

- An optimal solution with cost 1.75, improving on Slide 18 which has a cost of 1.80. The key k_5, which has highest probability, is closer to the root.

Brute Force Approach

- A brute force approach would enumerate all possibles BSTs over $k_1, \ldots, k_n$, compute their costs, and return the smallest.
- It would take exponential time because there is an exponential number of such BSTs.
  - Proof?
- So we will try dynamic programming.

Structure of an Optimal BST



- The subtree rooted at k_4 contains keys k_3, k_4 and d_2, d_3, d_4.

Structure of an Optimal BST

Lemma

Every subtree is either a single dummy node, or it contains consecutive keys $k_i < k_{i+1} < \cdots < k_j$ and dummy nodes $d_{i-1}, d_i, \ldots, d_j$, for $1 \leqslant i \leqslant j \leqslant n$.

Proof.

Follows from Slide 6. □

Structure of an Optimal BST

- From the lemma above, each subtree of T is a subtree $T[i,j]$ where $j \geqslant i-1$ such that
  - $T[i,j]$ contains keys $k_i, \ldots, k_j$ and $d_{i-1}, \ldots, d_j$ if $i \leqslant j$.
  - $T[i,i-1]$ only contains the dummy key d_{i-1}.
- The *weight* of the subtree $T[i,j]$ is the sum of the probabilities of its nodes. So

$$w[i,j] = \sum_{\ell=i}^{j} p_\ell + \sum_{\ell=i-1}^{j} q_\ell \qquad \text{whenever } 1 \leqslant i \leqslant j \leqslant n$$

$$w[i,i-1] = q_{i-1} \qquad \text{for } i = 1, 2, \ldots, n+1$$

Structure of an Optimal BST

Lemma

If $1 \leqslant i \leqslant j \leqslant n$ *and* $T[i,j]$ *is a subtree of an optimal binary search tree* T*, then* $T[i,j]$ *must be optimal for the subproblem with keys* $k_i, \ldots, k_j$ *and cost function*

$$\text{cost}(T[i,j]) = \sum_{\ell=i}^{j} p_\ell \,\text{depth}_{T[i,j]}(k_\ell) + \sum_{\ell=i-1}^{j} q_\ell \,\text{depth}_{T[i,j]}(d_\ell).$$

Proof

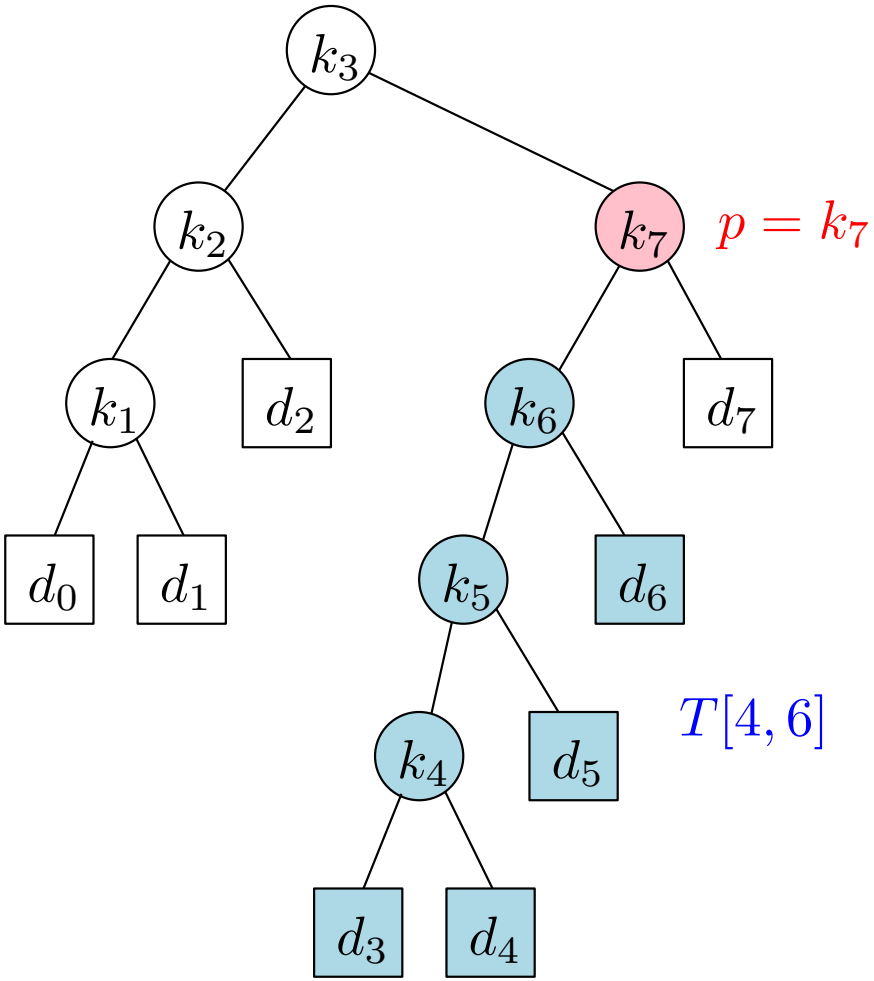


Figure: Tree T and subtree $T[4,6]$

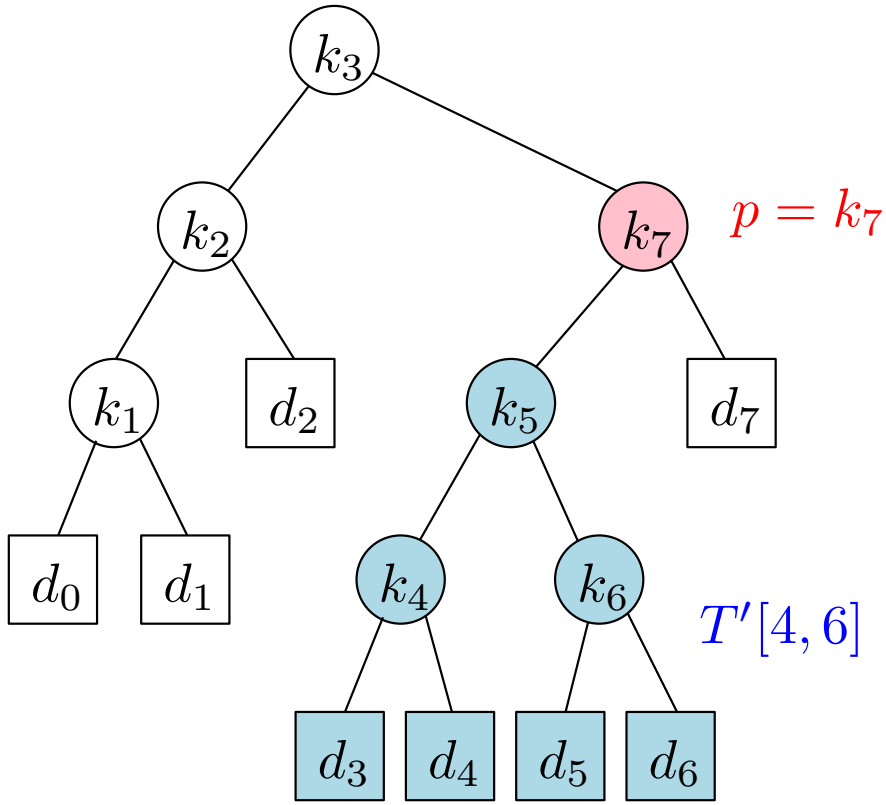


Figure: Tree T' obtained by replacing $T[4,6]$ with $T'[4,6]$

Proof

- We make a proof by contradiction.
- So we assume that $T[i,j]$ is not optimal.
- Then there is a subtree $T'[i,j]$ over the same set of keys such that

$$\text{cost}(T'[i,j]) < \text{cost}(T[i,j]).$$

- Let T' be the tree obtained from T by replacing $T[i,j]$ with $T'[i,j]$.
- Let p be the parent of the root of $T[i,j]$.
- Then p is the parent of the root of $T'[i,j]$.
- For every node v of $T[i,j]$

$$\text{depth}_T(v) = \text{depth}_T(p) + 1 + \text{depth}_{T[i,j]}(v)$$

- We denote $\delta = \text{depth}_T(p) + 1$.

Proof

- Then the contributions of the nodes of $T[i,j]$ to $\text{cost}(T)$ is

$$\sum_{\ell=i}^{j} p_\ell \,\text{depth}_T(k_\ell) + \sum_{\ell=i-1}^{j} q_\ell \,\text{depth}_T(d_\ell)$$

$$= \sum_{\ell=i}^{j} p_\ell \left(\delta + \text{depth}_{T[i,j]}(k_\ell)\right) + \sum_{\ell=i-1}^{j} q_\ell(\delta + \text{depth}_{T[i,j]}(d_\ell))$$

$$= \delta \cdot \left(\sum_{\ell=i}^{j} p_\ell + \sum_{\ell=i-1}^{j} q_\ell\right) + \text{cost}(T[i,j])$$

$$= \delta \cdot w[i,j] + \text{cost}(T[i,j])$$

Proof

- As the other nodes do not move, we have

$$\begin{aligned} \text{cost}(T') &= \text{cost}(T) + \text{cost}(T'[i,j]) - \text{cost}(T[i,j]) \\ &< \text{cost}(T). \end{aligned}$$

- It contradicts the lemma hypothesis that T is optimal.

Recursive Solution

- Let $c[i,j] = \text{cost}(T[i,j])$ be the cost of an optimal subtree $T[i,j]$ where $j \geqslant i-1$.
- As $T[i,i-1]$ has only one node with depth 0, we have $c[i,i-1] = 0$.
- If k_r is the root of an optimal subtree containing $k_i, \ldots, k_j$, then we have

$$c[i,j] = c[i,r-1] + w[i,r-1] + c[r+1,j] + w[r+1,j] \quad (2)$$

because for each node v of a subtree $T[i,r-1]$

$$\text{depth}_{T[i,j]}(v) = 1 + \text{depth}_{T[i,r-1]}(v),$$

and for each node v of a subtree $T[r+1,j]$

$$\text{depth}_{T[i,j]}(v) = 1 + \text{depth}_{T[r+1,j]}(v).$$

Recursive Solution

As we seek to minimize $c[i, j]$, it follows from Equation (2) that $c[i, j]$ is given by the relation

$$c[i, j] = \min_{i \leqslant r \leqslant j} (c[i, r-1] + w[i, r-1] + c[r+1, j] + w[r+1, j]) \qquad (3)$$

whenever $1 \leqslant i \leqslant j \leqslant n$.

As mentioned in previous slide, we also have

$$c[i, i-1] = 0$$

whenever $1 \leqslant i \leqslant n+1$.

Dynamic Programming

- Equation 3 requires to know the weight $w[i, j]$ of each possible subtree.
- We first compute it in time $\Theta(n^2)$.

Pseudocode

```
1: procedure ComputeWeights(p, q, n)
2:     w[1...n+1][0...n] ← new array
3:     for i ← 1, n+1 do
4:         w[i, i−1] ← q[i−1]
5:         for j ← i, n do
6:             w[i, j] ← w[i, j−1] + p_j + q_j
7:     return w
```

Dynamic Programming

- Equation 3 shows how to compute the optimal cost $c[i, j]$ as a function of $c[i, r-1]$ and $c[r+1, j]$ where $i \leqslant r \leqslant j$.
- It means that $c[i, j]$ only depends on smaller intervals, that is, we only need costs $c[i', j']$ such that $j' - i' < j - i$.
- So we will compute $c[i, j]$ by increasing length $\ell = j - i + 1$.
- Pseudocode on next slide. Comments:
  - The array w in Line 1 is the output of the procedure on Slide 33.
  - Lines 3–4 set cost 0 to all subtrees formed by a single leaf. (See Slide 25.)
  - The outer loop (Lines 5–12) goes by increasing size $\ell = j - i + 1$ of the subproblem.
  - Line 8–12 implement Equation 3.

Dynamic Programming

Pseudocode

```
1:  procedure OptimalCost(p, q, n, w)
2:      c[1...n+1, 0...n] ← new array
3:      for i ← 1, n+1 do
4:          c[i, i−1] ← 0
5:      for ℓ ← 1, n do
6:          for i ← 1, n−ℓ+1 do
7:              j ← i+ℓ−1
8:              c[i, j] ← ∞
9:              for r ← i, j do
10:                 t ← c[i, r−1] + w[i, r−1] + c[r+1, j] + w[r+1, j]
11:                 c[i, j] ← min(t, c[i, j])
12:     return c[1,n]
```

Analysis

- It is easy to see that the running time is $O(n^3)$ because of the three nested loops that are iterated at most n times each.
- As each subproblem $T[i,j]$ is solved in $\Theta(j-i+1)$ time, the total running time is $\Theta(f(n))$ where

$$f(n) = \sum_{1 \leqslant i \leqslant j \leqslant n} j - i + 1.$$

- On next slide, we argue that $f(n) = \Omega(n^3)$, which completes the proof that the running time is $\Theta(n^3)$.

Analysis

- First assume that n is a multiple of 4.

$$f(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} j - i + 1 \geqslant \sum_{i=1}^{n/4} \sum_{j=3n/4}^{n} j - i + 1$$

$$\geqslant \sum_{i=1}^{n/4} \sum_{j=3n/4}^{n} \frac{n}{2} \geqslant \frac{n}{4} \cdot \frac{n}{4} \cdot \frac{n}{2} = \frac{n^3}{32} = \Omega(n^3)$$

- If n is not a multiple of 4: Clearly f is an increasing function so

$$f(n) \geqslant f(4\lfloor n/4 \rfloor) \geqslant \frac{(4\lfloor n/4 \rfloor)^3}{32} \geqslant \frac{(n-4)^3}{32} = \Omega(n^3).$$

Computing an Optimal Solution

- We only computed the optimal *cost* of a BST.
- We did not show how to compute the optimal BST itself.
- As usual, it can be done quickly using the tables that we computed. (Left as an exercise.)
- Next slide shows a modification of the pseudocode that allows to easily recover an optimal BST by constructing an auxiliary table.
- The reconstruction procedure is left as an exercise.

Computing an Optimal Solution

Pseudocode

```
1:  procedure OptimalCost2(p, q, n, w)
2:      c[1...n+1, 0...n] ← new array
3:      r[1...n, 1...n] ← new array
4:      for i ← 1, n+1 do
5:          c[i, i−1] ← 0
6:      for ℓ ← 1, n do
7:          for i ← 1, n−ℓ+1 do
8:              j ← i+ℓ−1
9:              c[i, j] ← ∞
10:             for r ← i, j do
11:                 t ← c[i, r−1] + w[i, r−1] + c[r+1, j] + w[r+1, j]
12:                 if t < c[i, j] then
13:                     c[i, j] ← t
14:                     r[i, j] ← r
15:     return arrays c and r
```

Concluding Remarks

- The brute force approach takes exponential time, and dynamic programming takes cubic time.
- So once again dynamic programming improved the running time from exponential to polynomial.
- A running time of $\Theta(n^3)$ suggests that we can solve the problem up to roughly $n = 1000$. The brute force approach probably fails for n less than 50, as we saw in the case of binomial coefficients.
- Difference with the textbook: The textbook uses

$$e[i, j] = c[i, j] + w[i, j].$$

in the dynamic programming approach.