CSE331 Introduction to Algorithm
Lecture 7: Quicksort

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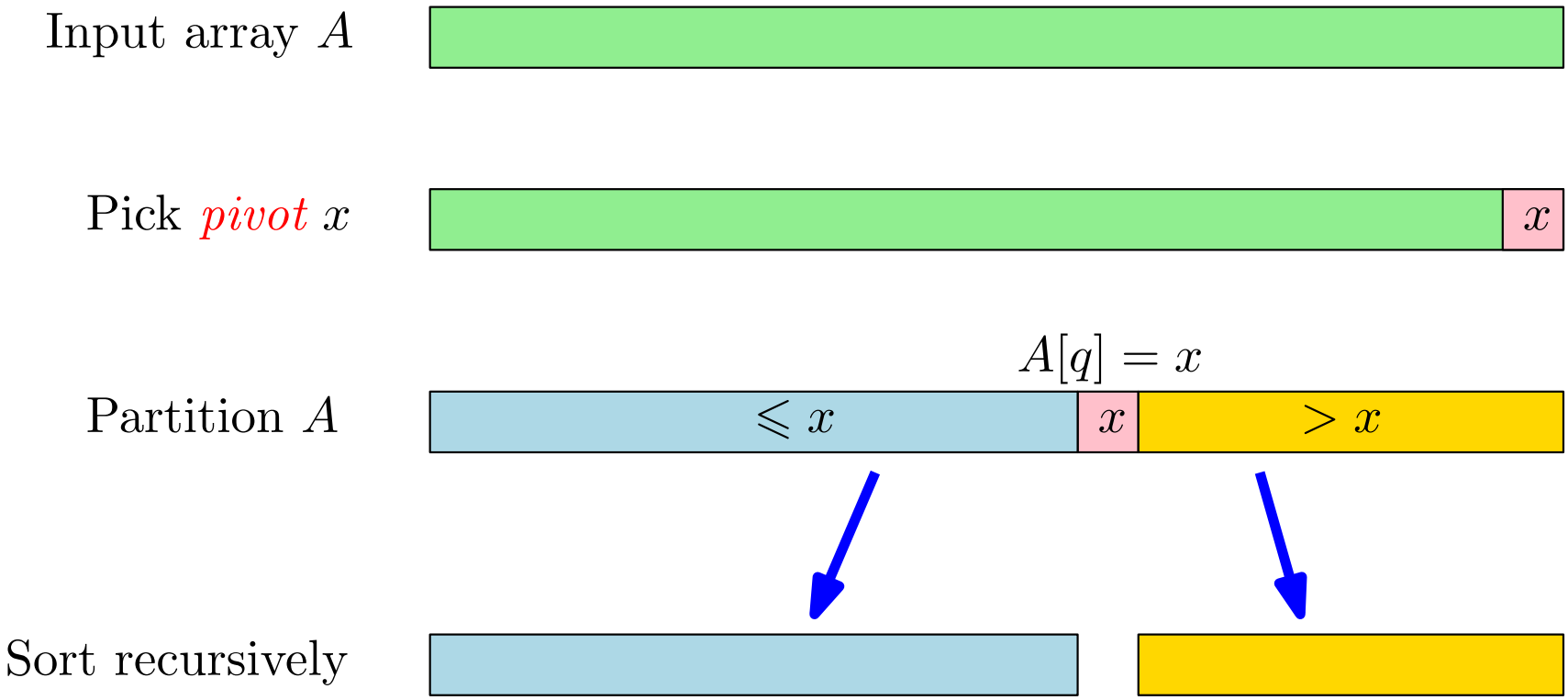
Introduction

- Today, another sorting algorithm: QUICKSORT.
- A divide and conquer algorithm, but different from MERGE SORT
- An important algorithm as it is used in practice.
- Easy to implement, but analysis is difficult.
- Reference: Chapter 7 of the textbook Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.

Quicksort: Overview



Quicksort: Overview

- If the pivot x falls in the middle (i.e. $q = \lfloor n/2 \rfloor$), then the running time satisfies $T(n) = 2T(n/2) + \Theta(n)$.
- So $T(n) = \Theta(n \log n)$. (Same as Merge Sort.)
- This is the best case scenario. Unfortunately there is no simple way of doing it.
- We will see that a *random* choice for the pivot is a good option.

Quicksort

Pseudocode

```
1: procedure QUICKSORT(A[p ... r])
2:     if p < r then
3:         q ← PARTITION(A[p ... r])
4:         QUICKSORT(A[p ... q − 1])
5:         QUICKSORT(A[q + 1 ... r])
```

- In the next few slides, we describe two approaches for the Partition procedure.

Partitioning the Array

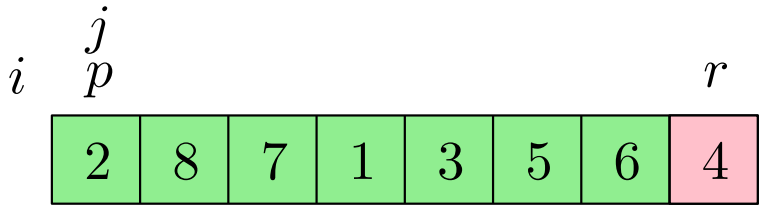
Pseudocode

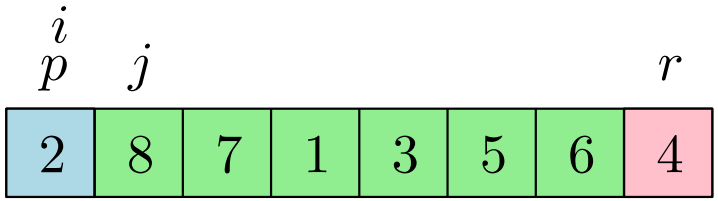
```
1:  procedure NaivePartition(A[p ... r])
2:      x ← A[r]                                  ▷ Choose the pivot
3:      L, R ← new arrays
4:      j ← 1, k ← 1
5:      for i = p, r − 1 do                       ▷ Fill arrays L and R
6:          if A[i] ⩽ x then
7:              L[j] ← A[i]
8:              j ← j + 1
9:          else
10:             R[k] ← A[i]
11:             k ← k + 1
12:     A[p, p + j − 2] ← L[1, j − 1]             ▷ Copy L into A
13:     A[p + j − 1] ← x                          ▷ Place the pivot
14:     A[p + j, r] ← R[1, k − 1]                 ▷ Copy R into A
15:     return p + j − 1
```

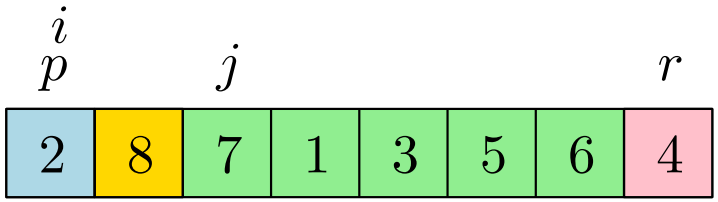
Partitioning the Array

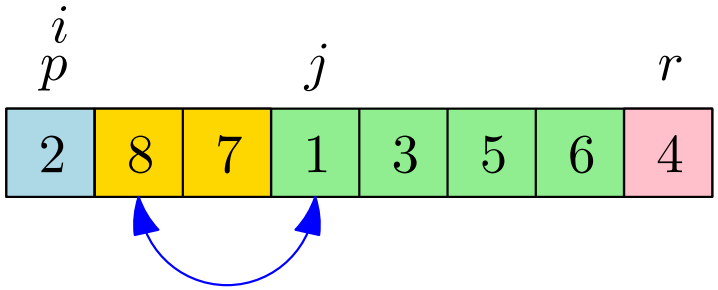
- The naive approach above runs in linear time.
- The pivot is chosen to be $A[r]$, but this is not a strong restriction. We could pick it anywhere.
- Drawback of the naive approach:
  - It fills up two auxiliary arrays.
  - i.e. it is not *in place*.
- Next two slides: a shorter, in-place version, that also runs in linear time.

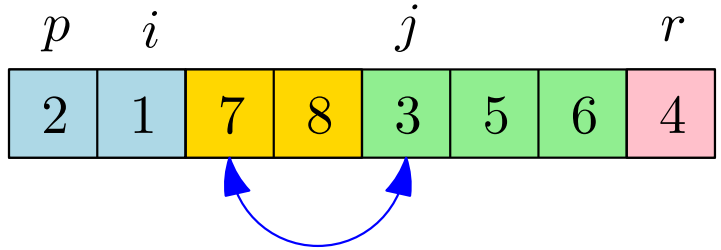
Partitioning the Array: Example

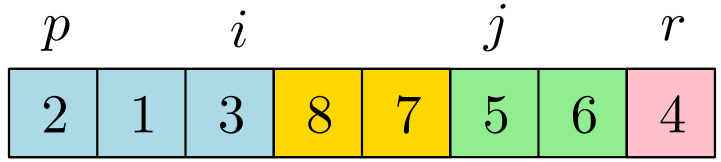


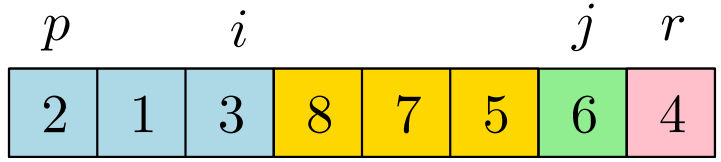


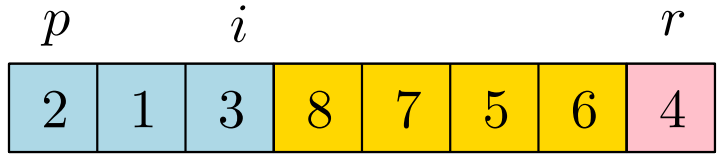


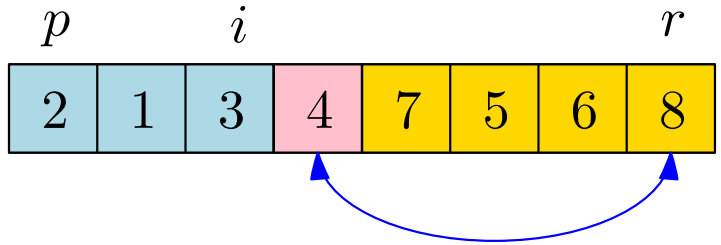












Partitioning the Array

Pseudocode

```
1: procedure PARTITION(A[p ... r])
2:     x ← A[r]                                  ▷ Choosing the pivot
3:     i ← p − 1
4:     for j ← p, r − 1 do
5:         if A[j] ⩽ x then
6:             i ← i + 1
7:             Exchange A[i] with A[j]
8:     Exchange A[i + 1] with A[r]
9:     return i + 1                              ▷ Returns the position of the pivot
```

Proof of Correctness

Loop invariants

At the beginning of each iteration of the loop, for any array index k,

1. If $p \leqslant k \leqslant i$, then $A[k] \leqslant x$.
2. If $i < k < j$, then $A[k] > x$.
3. If $k = r$, then $A[k] = x$.

- We need to prove Initialization, Maintenance, and Termination.
- Done in class. See textbook p. 173

Worst-Case Analysis

- The partition procedure runs in $\Theta(n)$ time.
- So the worst case running time $T(n)$ of QUICKSORT satisfies the relation:

$$T(n) \leqslant \max_{0 \leqslant q \leqslant n-1} (T(q) + T(n-q-1)) + \Theta(n).$$

- Guess: $T(n) = \Theta(n^2)$.
- Using the substitution method, we prove that, $T(n) = O(n^2)$.
  - Done in class. See textbook p. 180.
- Example where $T(n) = \Omega(n^2)$:

$$A[1 \ldots n] = [1, 2, \ldots, n],$$

so QUICKSORT performs badly when the input is already sorted.

Randomized Quicksort

- In previous slide, we saw that Quicksort performs badly when the pivot is the largest element in the array.
- One way to avoid it is to pick a *random* pivot.

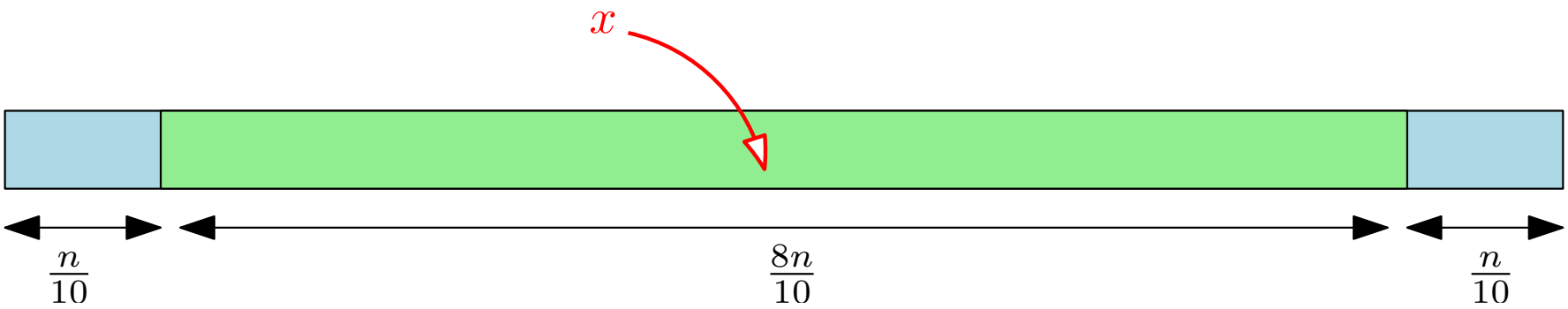
Pseudocode

```
1: procedure RandomizedQuicksort(A[p ... r])
2:     if p < r then
3:         i ← Random(p, r)
4:         Exchange A[r] with A[i]
5:         q ← Partition(A[p ... r])
6:         RandomizedQuicksort(A[p ... q − 1])
7:         RandomizedQuicksort(A[q + 1 ... r])
```

- Line 3: The integer i is chosen uniformly at random between p and r.

Randomized Quicksort

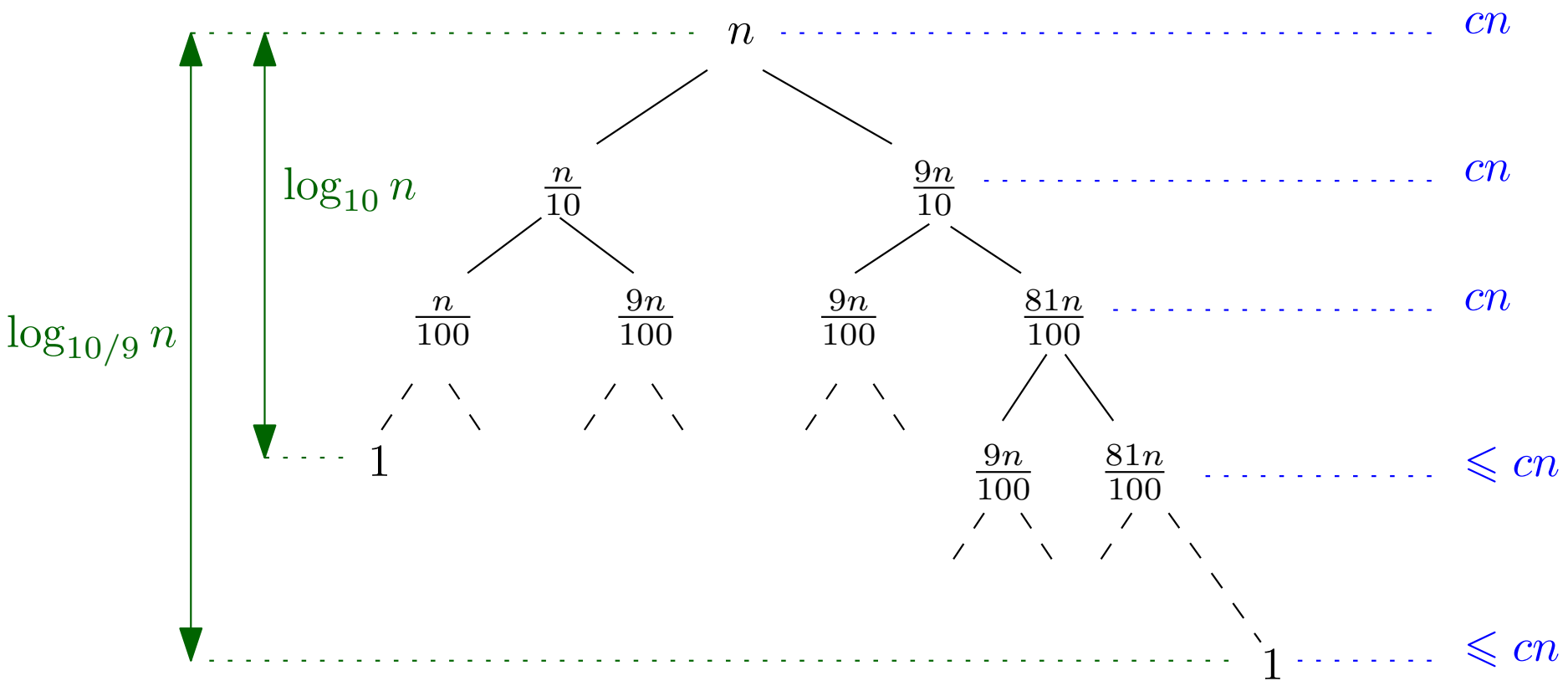
- As $A[r]$ is the pivot, the randomized Quicksort algorithm uses as pivot an element of the current array chosen uniformly at random.
- We now explain intuitively why the expected running time should be $O(n \log n)$.



- With probability 8/10, the pivot falls in the green area.
- So the array is split into subarrays of sizes $\geqslant n/10$.
- We now apply the recursion tree method to

$$T(n) = T(n/10) + T(9n/10) + \Theta(n).$$

Randomized Quicksort



Total: $\Theta(n \log n)$

Randomized Quicksort

- Every level of the tree has cost $\leqslant cn$.
- Each level is full until depth $\log_{10} n = \Theta(\log n)$.
- The height is $\log_{10/9} n = \Theta(\log n)$.
- So the running time is $\Theta(n \log n)$, even in this very unbalanced case.
- Conjecture: The *expected* running time of randomized Quicksort is $\Theta(n \log n)$.
  - Here, "expected" running time means the average running time over all random choices of pivots.
  - It is not an average running time over a distribution of input arrays.
- We will prove it rigorously in the next lecture, while studying a related algorithm.

C Code (source: The C Programming Language, by Kernighan & Ritchie)

```
void qsort(int v[], int left, int right){
   int i, last;
   void swap(int v[], int i, int j);
   if (left >= right)
      return;
   swap(v, left, (left + right)/2);
   last = left;
   for (i = left + 1; i <= right; i++)
      if (v[i] < v[left])
         swap(v, ++last, i);
   swap(v, left, last);
   qsort(v, left, last-1);
   qsort(v, last+1, right);
}
void swap(int v[], int i, int j){
   int temp; temp = v[i]; v[i] = v[j]; v[j] = temp;
}
```

Conclusion

- QUICKSORT is simple and practical, but its analysis is non-trivial.
- The randomized version runs in expected $O(n \log n)$ time *on any input*.
- It can be shown that this bound holds with high probability.
- It is often the case that simple randomized algorithms can be found for solving a given problem. It is an attractive choice for programmers as these algorithms are easy to implement and fast in practice.
- The deterministic version of QUICKSORT runs in quadratic time on some input, for instance sorted input.
- QUICKSORT is in-place.