

CSE331 Introduction to Algorithm

Lecture 5: Matrix Multiplication

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Introduction

- Reference: Sections 4.2 of the textbook [Introduction to Algorithms](#) by Cormen, Leiserson, Rivest and Stein.
- Today's topic: *Strassen's algorithm*, an efficient algorithm for computing matrix products.
- It is a divide and conquer algorithm.
- It splits the problem recursively into 7 subproblems.
- Its running time is $\Theta(n^{\log 7}) \simeq n^{2.81}$.
- Naive algorithm: $\Theta(n^3)$.
- Best known: $\simeq n^{2.37}$. Not covered in CSE331.
- Matrix multiplication algorithms are very important in numerical analysis, and also have applications to combinatorial algorithms.

Matrix Multiplication

- In this lecture, we only consider $n \times n$ square matrices where n is a power of 2.

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$$

Matrix Multiplication

- Let $A = (a_{ij})$, $B = (b_{ij})$ be $n \times n$ matrices.
- Then the product $C = A \cdot B$ is the $n \times n$ matrix $C = (c_{ij})$ such that

$$c_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}.$$

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & c_{ij} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \cdot \begin{pmatrix} \dots & \dots & b_{1j} & \dots & \dots \\ \dots & \dots & b_{2j} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & b_{nj} & \dots & \dots \end{pmatrix}$$

- So c_{ij} is the dot product of the i th row of A with the j th column of B .

Naive Algorithm

Pseudocode

```
1: procedure NAIVE SQUARE MATRIX MULTIPLICATION( $A, B$ )
2:    $n \leftarrow$  size of  $A$ 
3:    $C \leftarrow$  new  $n \times n$  matrix
4:   for  $i \leftarrow 1, n$  do
5:     for  $j \leftarrow 1, n$  do
6:        $c_{ij} \leftarrow 0$ 
7:       for  $k \leftarrow 1, n$  do
8:          $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$ 
9:   return  $C$ 
```

- Running time?
- Dominated by lines 7 and 8, which are executed n^3 times.
- So it is $\Theta(n^3)$.
 - ▶ Remark: Here n is not the input size. The input size is $2n^2$.

Block Matrices

- Example:

$$A = \begin{pmatrix} 2 & 5 & 3 & 1 \\ 4 & 3 & 2 & 2 \\ 3 & 1 & 5 & 6 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & 5 & 3 & 1 \\ 4 & 3 & 2 & 2 \\ \hline 3 & 1 & 5 & 6 \\ 1 & 3 & 2 & 4 \end{array} \right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where

$$A_{11} = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 5 & 6 \\ 2 & 4 \end{pmatrix}$$

Block Matrices

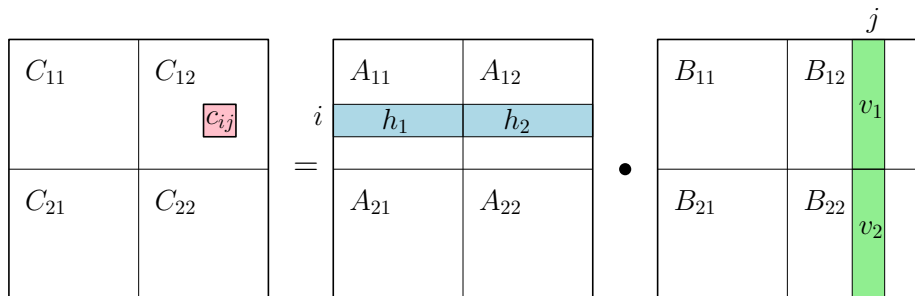
Definition

A *block matrix* is a matrix which is interpreted as having been broken into sections called *blocks* or *submatrices*.

- In this lecture, we will only consider blocks of size $(n/2) \times (n/2)$ from an $n \times n$ matrix:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Block Matrix Multiplication



$$c_{ij} = h_1 \cdot v_1 + h_2 \cdot v_2$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$

Block Matrix Multiplication

- Suppose $C = A \cdot B$ are $n \times n$ matrices.

$$\begin{aligned} C &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{pmatrix} \end{aligned}$$

- So the product of two block matrices can be done block by block, in the same way as we multiply two 2×2 matrices.
- I will not prove it formally, but the previous slide gives the intuition.

First Divide-and-Conquer Algorithm

Pseudocode

```
1: procedure MULTIPLY( $A, B$ )
2:    $n \leftarrow$  size of  $A$ 
3:    $C \leftarrow$  new  $n \times n$  matrix
4:   if  $n = 1$  then
5:      $c_{11} \leftarrow a_{11} \cdot b_{11}$ 
6:     return  $C$ 
7:   partition  $A, B, C$  into four  $(n/2) \times (n/2)$  blocks each
8:    $C_{11} \leftarrow$  Multiply( $A_{11}, B_{11}$ ) + Multiply( $A_{12}, B_{21}$ )
9:    $C_{12} \leftarrow$  Multiply( $A_{11}, B_{12}$ ) + Multiply( $A_{12}, B_{22}$ )
10:   $C_{21} \leftarrow$  Multiply( $A_{21}, B_{11}$ ) + Multiply( $A_{22}, B_{21}$ )
11:   $C_{22} \leftarrow$  Multiply( $A_{21}, B_{12}$ ) + Multiply( $A_{22}, B_{22}$ )
12:  return  $C$ 
```

First Divide-and-Conquer Algorithm: Analysis

- Ignoring recursive calls, lines 8–11 take $\Theta(n^2)$ time.
 - ▶ Consists of 4 matrix addition, which take $\Theta(n^2)$ time each.
- Line 7:
 - ▶ We have not explained in details how matrices are split.
 - ▶ We can create 12 new $(n/2) \times (n/2)$ arrays and copy the entries of A , B and C into them.
 - ▶ It takes time $\Theta(n^2)$.
 - ▶ The textbook mentions a better approach, but in any case it will not improve the time bound as lines 8–11 already take $\Theta(n^2)$.
- Line 12:
 - ▶ copy $C_{11}, C_{12}, C_{21}, C_{22}$ into C which takes $O(n^2)$ time.
- After counting the 8 recursive calls, we obtain the recurrence relation:

$$T(n) = 8T(n/2) + \Theta(n^2)$$

First Divide-and-Conquer Algorithm: Analysis

- We apply the Master theorem.
- $a = 8$, $b = 2$, and $f(n) = cn^2$ for some constant c .
- $n^{\log_b a} = n^3$.
- We are in Case 1: $f(n) = O(n^{2.99})$.
- So $T(n) = \Theta(n^3)$.
- It is not faster than the naive algorithm.
- Conclusion: divide and conquer does not always help.

Strassen's Algorithm

- In the analysis from previous slide, if $a = 7$ instead of $a = 8$, we get an improved running time $\Theta(n^{\log 7}) \simeq n^{2.81}$.
- $a = 8$ comes from the 8 recursive matrix multiplications.
- Matrix additions do not matter as they take $\Theta(n^2)$ time.
- Idea: Try a similar approach with only 7 recursive multiplications.

Strassen's Algorithm

Strassen's Algorithm

- 1 Divide input matrices A, B and output matrix C into $(n/2) \times (n/2)$ blocks.
- 2 Compute 7 $(n/2) \times (n/2)$ matrices P_1, \dots, P_7 from the submatrices A_{ij}, B_{ij} , using only one recursive multiplication for each, as well as one or two additions or subtractions. (See Slide 16, left.)
- 3 Compute $C_{11}, C_{12}, C_{21}, C_{22}$ using only additions and subtractions. (See Slide 16, right.)

Strassen's Algorithm

Intermediate results

$$P_1 = A_{11} \cdot (B_{12} - B_{22})$$

$$P_2 = (A_{11} + A_{12}) \cdot B_{22}$$

$$P_3 = (A_{21} + A_{22}) \cdot B_{11}$$

$$P_4 = A_{22} \cdot (B_{21} - B_{11})$$

$$P_5 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$

$$P_6 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$$

$$P_7 = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})$$

Final result

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

$$C_{12} = P_1 + P_2$$

$$C_{21} = P_3 + P_4$$

$$C_{22} = P_5 + P_1 - P_3 - P_7$$

Strassen's Algorithm: Correctness

- We need to verify that the algorithm is correct.
- So we need to prove that the expressions for C_{ij} on Slide 16 are consistent with Slide 10.
- We need to prove that

$$P_5 + P_4 - P_2 + P_6 = A_{11}B_{11} + A_{12}B_{21}$$

$$P_1 + P_2 = A_{11}B_{12} + A_{12}B_{22}$$

$$P_3 + P_4 = A_{21}B_{11} + A_{22}B_{21}$$

$$P_5 + P_1 - P_3 - P_7 = A_{21}B_{12} + A_{22}B_{22}$$

Strassen's Algorithm: Correctness

- We only prove the first equation

$$P_5 + P_4 - P_2 + P_6 = A_{11}B_{11} + A_{12}B_{21}.$$

The other three proofs are analogous.

$$\begin{array}{r} P_5 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22} \\ P_4 = \phantom{A_{11}B_{11}} \phantom{+ A_{11}B_{22}} - A_{22}B_{11} \phantom{+ A_{22}B_{22}} + A_{22}B_{21} \\ - P_2 = \phantom{A_{11}B_{11}} - A_{11}B_{22} \phantom{+ A_{22}B_{11}} \phantom{+ A_{22}B_{22}} - A_{12}B_{22} \\ P_6 = \phantom{A_{11}B_{11}} \phantom{+ A_{11}B_{22}} \phantom{- A_{22}B_{11}} - A_{22}B_{22} - A_{22}B_{21} + A_{12}B_{22} + A_{12}B_{21} \\ \hline C_{11} = A_{11}B_{11} \phantom{+ A_{11}B_{22}} \phantom{- A_{22}B_{11}} \phantom{+ A_{22}B_{21}} \phantom{- A_{12}B_{22}} + A_{12}B_{21} \end{array}$$

Conclusion

- The running time of Strassen's algorithm is given by the recurrence relation

$$T(n) = 7T(n/2) + \Theta(n^2)$$

- By the master theorem, it yields $T(n) = \Theta(n^{\log_2 7}) \simeq n^{2.81}$.
- It improves on the naive approach which runs in $\Theta(n^3)$ time.
- Strassen's algorithm, published in 1969, was the first known algorithm to run in $o(n^3)$ time.
- The algorithm and its proof are simple, the difficult part is to come up with the matrices P_1, \dots, P_7 .
- Apparently it is not known how Strassen found these matrices.
- It cannot be done with 6 matrices:
On multiplications of 2×2 matrices by Winograd, 1971.
- Faster algorithms use a similar approach, but are much more complicated.